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ON INTERNAL GRAVITY WAVES GENERATED BY LOCAL DISTURBANCES

By

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NOTATION

A	Source amplitude
b	Exponent of density stratification
c	Phase velocity
g	Acceleration due to gravity
G	Magnitude of group velocity
Gx, Gy	Components of group velocity
$H_\nu^{(1)}$	Hankel function of the first kind of order ν
k	Magnitude of wave number
K_ν	Modified Bessel function of the second kind of order ν
M	$M^2 = \frac{N^2}{\omega^2} - 1$
N	Vaisala frequency
P	$P^2 = - M^2$
$p, p', \bar{p}$	Total, perturbation, and mean pressure, respectively
q	Source distribution
t	Time
u', v'	Components of perturbation velocity
u, v	Components of total velocity

V	$V = \sqrt{\bar{\rho}}\ v'$
x,y	Cartesian coordinates
ω	Frequency
α,β	Components of wave number
$\rho,\rho',\bar{\rho}$	Total, perturbation,and mean density, respectively
θ	Direction of wave number
Φ	Amplitude of V
ϕ	Green's function
$\delta(x)$	Dirac delta function

INTRODUCTION

Love (see Lamb 1945), was the first to derive the linearized equations governing small oscillations about the state of hydrostatic equilibrium in a continuously stratified fluid. The propagation of free elementary waves in continuously stratified fluids were studied by Yih (1960), Eckart (1960), Yanowitch (1962) and Tolstoy (1963). Some of these authors also considered the effects of compressibility and rotation.

In this report the internal waves generated by a time dependent local disturbance in an incompressible continuously stratified fluid are examined. Following the Love theory of wave motion in heterogeneous liquids, the motion is regarded as a small perturbation about the state of hydrostatic equilibrium. For simplicity, the equilibrium density profile is taken to be exponential whereby the coefficients of the governing equations become constants. The complications of multiple reflections from physical boundaries are avoided by considering a fluid of infinite extent. The disturbance is taken to be two-dimensional and time dependent but non-moving.

THE GOVERNING EQUATIONS

Consider the equations governing the two-dimensional motion of an incompressible, continuously stratified fluid under gravity. Let (x,y) be a Cartesian coordinate system oriented with gravity pointing in the negative y direction, and (u,v) be the velocity components in the x and y directions respectively.

Since the fluid is incompressible, then

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 2\pi q(x,y,t) \qquad [1]$$

where $2\pi q$ denotes the volume rate of outflow from a source distribution in the fluid. Conservation of mass demands that

$$\frac{\partial \rho}{\partial t} + \rho \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} = 2\pi q(x,y,t)\rho_*(x,y,t) \qquad [2]$$

where ρ is the density field and ρ_* is the density of the fluid being given out (in this case, ρ_* is left open) or taken in (in this case, ρ_* must be identically equal to the ambient density) by the source distribution. Subtracting [1] from [2] yields the following 'equation of state':

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} = 2\pi q(\rho_* - \rho)H(q) \qquad [3]$$

where H denotes the Heaviside function. If ρ_* is controlled such that it is always equal to the ambient density, then Equation [3] simplifies to

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} = 0 \qquad [4]$$

In what follows, this is assumed to be always the case.

Equations [1] and [4], together with the Euler equations

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = - \frac{1}{\rho} \frac{\partial p}{\partial x} \qquad [5]$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = - \frac{1}{\rho} \frac{\partial p}{\partial y} - g \qquad [6]$$

where p denotes the pressure and g, the acceleration due to gravity, form a system of four equations for the four unknowns (u, v, p, ρ).

If the motion is weak, it may be regarded as a small perturbation about the state of hydrostatic equilibrium and, as a first approximation, second order effects may be neglected. Denoting the perturbations by (u', v', p', ρ'), the hydrostatic pressure by $\bar{p}$ and the equilibrium density by $\bar{\rho}$, Equations [1], [4], [5] and [6], on neglecting second order terms, become

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = 2\pi q \qquad [7]$$

$$\frac{\partial \rho'}{\partial t} + v\frac{d\bar{\rho}}{dy} = 0$$

$$\bar{\rho}\,\frac{\partial u'}{\partial t} = -\frac{\partial p'}{\partial x} \qquad [9]$$

$$\bar{\rho}\,\frac{\partial v'}{\partial t} = -\frac{\partial p'}{\partial y} - g\rho' \qquad [10]$$

The coefficients are seen to be functions of y only and the resulting system is linear. From this system of equations, the following equation for v' may be uncoupled:

$$\frac{\partial^2}{\partial t^2}\left\{\frac{\partial^2}{\partial y^2}(\bar{\rho}v') + \frac{\partial^2}{\partial x^2}(\bar{\rho}v') + \frac{N^2}{g}\frac{\partial}{\partial y}(\bar{\rho}v') + \frac{2N}{g}\frac{dN}{dy}\bar{\rho}v'\right\}$$

$$+ N^2\frac{\partial^2}{\partial x^2}(\bar{\rho}v') = 2\pi\frac{\partial}{\partial y}\left(\bar{\rho}\,\frac{\partial^2 q}{\partial t^2}\right) \qquad [11]$$

where N is the Vaisala frequency defined by

$$N = \sqrt{-\frac{g}{\bar{\rho}}\frac{d\bar{\rho}}{dy}} \qquad [12]$$

This important parameter serves to characterize the capability of the medium as an internal wave carrier.

In what follows, attention is focused entirely on the vertical velocity component only. It is convenient to transform it to a new variable V according to

$$V = \sqrt{\bar{\rho}}\, v' \qquad [13]$$

The Equation for V is

$$\frac{\partial^2}{\partial t^2}\left\{\frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial x^2} + \left(\frac{N}{g}\frac{dN}{dy} - \frac{N^4}{4g^2}\right) V\right\} + N^2 \frac{\partial^2 V}{\partial x^2}$$

$$= \frac{2\pi}{\sqrt{\bar{\rho}}} \frac{\partial}{\partial y}\left(\bar{\rho}\,\frac{\partial^2 q}{\partial t^2}\right) \qquad [14]$$

For an exponential density profile, viz.

$$\bar{\rho} = \rho_o\, e^{-2by} \qquad [15]$$

where ρ_o and b are constants, Equation [14] simplifies to

$$\frac{\partial^2}{\partial t^2}\left\{\frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial x^2} - b^2 V\right\} + N^2 \frac{\partial^2 V}{\partial x^2}$$

$$= 2\pi\sqrt{\rho_o}\; e^{by} \frac{\partial}{\partial y}\left(e^{-2by} \frac{\partial^2 q}{\partial t^2}\right) \qquad [16]$$

and the Vaisala frequency $N = \sqrt{2gb}$ is constant throughout the fluid which means that the fluid is homogeneous in character as a wave carrier but anisotropic.

ELEMENTARY WAVES

In this section, certain results concerning the behaviour of elementary waves in a stratified fluid are discussed. For a more general treatment in this respect, the reader is referred to Eckart (1960).

The homogeneous character of the fluid suggests that plane waves of the form

$$V = A\, e^{i(\alpha x + \beta y - \omega t)}$$

where A is a complex constant, α, β and ω are real constants, are possible homogeneous solutions of Equation [16]. Substitution into Equation [16] reveals that this is indeed so provided that the following characteristic equation is satisfied, viz

$$\omega^2(\alpha^2 + \beta^2 + b^2) - N^2\alpha^2 = 0 \qquad [17]$$

In the first place, a necessary condition for Equation [17] to have real solutions is that ω must be less than the Vaisala frequency, i.e.

$$\frac{N^2}{\omega^2} - 1 = M^2 > 0$$

Since the Vaisala frequency measures the stability or 'stiffness' of the fluid, the above condition implies that higher frequency waves require a stiffer medium; a result which is in accord with intuition. For each ω that is less than the Vaisala frequency, Equation [17] is satisfied by all points lying on an hyperbola of two sheets in the wave number plane. In Figure 1 the hyperbolas in the first quadrant for various ω are shown. The family of curves is symmetric about both the vertical and horizontal axes. It is seen that, for a given ω/N, the magnitude, k, of the wave number depends on its inclination from the horizontal axis, θ, which cannot exceed $\tan^{-1} M$.

The phase velocity or, in other words, the velocity of propagation of the individual crests and troughs has the same direction as the wave number and its magnitude, c, is given by

$$c = \frac{N}{b} \cdot \sqrt{\cos^2\theta - \frac{\omega^2}{N^2}} \qquad [18]$$

Figure 2 is a polar plot of c against θ for various values of ω. The fact that c is a function of θ demonstrates the anisotropic character of the medium.

The group velocity, according to which energy is propagated, is given by

$$(Gx, Gy) = \left(\frac{\partial\omega}{\partial\alpha}, \frac{\partial\omega}{\partial\beta}\right) \qquad [19]$$

The direction of the group velocity, then, is normal to that hyperbola in which the wave number is found at (α, β) and is, in general, different from the direction of the phase velocity. This behaviour is characteristic of wave motions in anisotropic media (see Lighthill 1960). At $\theta = 0$ the two velocities have the same direction, as θ increases the difference in direction increases and tends to $\pi/2$ as θ tends to $\tan^{-1} M$. The magnitude of the group velocity, G, is found to be

$$G = \frac{c}{as\ \theta} \frac{\omega^2}{N^2} \sqrt{M^2 + \tan^2\theta} \qquad [20]$$

Figure 3 is a polar plot of G against θ for various values of ω.

It is noted, that the amplitude of the actual velocity, v', is not constant but varies with y as $1/\sqrt{\bar{\rho}}$. It follows that the first order kinetic energy is conserved.

INTERNAL WAVES GENERATED BY AN OSCILLATING SINGULARITY

Consider the waves generated by a source of constant amplitude A and simple harmonic time dependency of frequency ω located at the origin. After a sufficient time has elapsed for the transients to die away, the dependent variables would also acquire the same time dependency. Therefore,

$$V(x,y,t) = \Phi(x,y)e^{-i\omega t} \qquad [21]$$

The equation governing the amplitude function $\Phi(x,y)$ is

$$\frac{\partial^2\Phi}{\partial y^2} - M^2 \frac{\partial^2\Phi}{\partial x^2} - b^2\Phi = 2\pi A \sqrt{\rho_o} \quad \delta(x)[\dot{\delta}(y) - b\delta(y)] \qquad [22]$$

where δ denotes the Dirac delta function and a dot denotes a differentiation.

First, consider the Green's function, Φ, of Equation [22]. By definition, Φ satisfies the following equation:

$$\frac{\partial^2\phi}{\partial y^2} - M^2 \frac{\partial^2\phi}{\partial x^2} - b^2\phi = \delta(x)\delta(y) \qquad [23]$$

The two dimensional Fourier integral representation of ϕ is easily found to be

$$\phi(x,y) = \frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{e^{i(\alpha x+\beta y)}}{M^2\alpha^2 - (\beta^2+b^2)}\, d\alpha d\beta \qquad [24]$$

Consider first the case where ω is larger than the Vaisala frequency, thus M^2 is real and negative. Let $M^2 = -P^2$ where P is real and positive. Rewrite Equation [24] as

$$\phi = -\frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{e^{i(\alpha x+\beta y)}}{\alpha^2P^2 + (\beta^2+b^2)}\, d\alpha d\beta \qquad [25]$$

Integrate first with respect to α. The integrand has simple poles at

$$\alpha_{\pm} = \pm\frac{i\sqrt{\beta^2+b^2}}{P} \qquad [26]$$

The contour is closed by a large semi-circle in the upper or lower half-plane depending on whether x is positive or negative and the integral evaluated by the method of residues; the contribution from the semi-circle vanishes as its radius tends to infinity. From here on, attention is focussed on $x > 0$ only. For $x > 0$

$$\phi = -\frac{1}{4\pi P}\int_{-\infty}^{+\infty} \frac{e^{i\beta y - \sqrt{\beta^2 + b^2}\,\frac{x}{P}}}{\sqrt{\beta^2 + b^2}}\, d\beta$$

$$= -\frac{1}{2\pi P}\int_{0}^{\infty} \frac{e^{-\frac{x}{P}\sqrt{\beta^2 + b^2}}}{\sqrt{\beta^2 + b^2}} \cos \beta y \, d\beta$$

But

$$\int_{0}^{\infty} \frac{e^{-\frac{x}{P}\sqrt{\beta^2 + b^2}}}{\sqrt{\beta^2 + b^2}} \cos \beta y \, d\beta = K_o\left[b\left(\frac{x^2}{P^2} + y^2\right)\right]$$

(see Watson 1958), where K_ν denotes the modified Bessel function of the second kind of order ν. Therefore,

$$\phi = -\frac{1}{2\pi P} K_o \left(b\sqrt{\frac{x^2}{P^2} + y^2} \right) \qquad [27]$$

Next, consider the case where ω is less than the Vaisala frequency, thus M^2 is real and positive. The poles now lie on the real α axis and the integral is indeterminate. Under such circumstances it is usual to deform the path of integration slightly such that it goes around instead of through the poles. How the path should be deformed depends on additional physical information. In the present problem it is physically obvious that ϕ must be symmetric in x and that all internal waves are outgoing waves from the source at the origin. With this considerations in mind it is found that the path of integration should be displaced above the negative pole and below the positive pole by small semi-circles. The integral is then evaluated as in the previous case to give

$$\phi = \frac{1}{4\pi M} \int_{-\infty}^{+\infty} \frac{e^{i(\beta y + \sqrt{\beta^2 + b^2}\, \frac{x}{M})}}{\sqrt{\beta^2 + b^2}} \, d\beta$$

$$= \frac{1}{2\pi M} \int_{0}^{\infty} \frac{e^{i \frac{x}{M} \sqrt{\beta^2 + b^2}}}{\sqrt{\beta^2 + b^2}} \cos \beta y \, d\beta$$

But

$$\int_0^\infty \frac{e^{i\frac{x}{M}\sqrt{\beta^2+b^2}}}{\sqrt{\beta^2+b^2}} \cos \beta y \, d\beta = \frac{\pi i}{2} H_o^{(1)}\left(b\sqrt{\frac{x^2}{M^2} - y^2}\right)$$

(see Watson 1958), where $H_\nu^{(1)}$ denotes the Hankel function of the first kind of order ν. Therefore,

$$\phi = - \frac{1}{4M} H_o^{(1)}\left(b\sqrt{\frac{x^2}{M^2} - y^2}\right) \qquad [28]$$

The motion generated by the oscillating source can be expressed in terms of ϕ as

$$v'(x,y,t) = e^{-i\omega t} 2\pi A \; e^{by}\left(\frac{\partial\phi}{\partial y} - b\phi\right) \qquad [29]$$

First examine the case where ω is greater than the Vaisala frequency. Substituting Equation [27] into [29] the following equation for v' is obtained

$$v' = e^{-i\omega t}\,\frac{Ab\,e^{by}}{P}\cdot\left[\frac{y}{\sqrt{\frac{x^2}{P^2}+y^2}}\,K_1\left(b\sqrt{\frac{x^2}{P^2}+y^2}\right)\right.$$

$$\left.+\,K_o\left(b\sqrt{\frac{x^2}{P^2}+y^2}\right)\right] \qquad [30]$$

As the stratification tends to zero, i.e. as $b \rightarrow 0$

$$v' \rightarrow e^{-i\omega t}\,A\,\frac{y}{x^2+y^2} \qquad [31]$$

This limit is the motion generated by an oscillating source in a homogeneous fluid. For finite b, the standing wave character of the motion is preserved because both K_o and K_1 are real for real arguments. However, the purely antisymmetric character of the vertical velocity component with respect to y is destroyed. It is noted that no progressive internal wave is generated. This is in accord with the result of the last section which excludes the possibility of progressive waves whose frequency is greater than the Vaisala frequency.

Next consider the case where ω is less than the Vaisala frequency. Substitution of Equation [28] into [29] results in the following equation for v':

$$v' = -e^{-i\omega t} \frac{\pi A b\, e^{by}}{2M} \left[\frac{y}{\sqrt{\frac{x^2}{M^2} - y^2}} H_1^{(1)}\left(b\sqrt{\frac{x^2}{M^2} - y^2}\right) - H_o^{(1)}\left(b\sqrt{\frac{x^2}{M^2} - y^2}\right) \right] \qquad [32]$$

The solution behaves very differently depending on whether $x^2/M^2 - y^2$ is positive or negative. Consider first the region which satisfies the inequality

$$\frac{x^2}{M^2} - y^2 < 0$$

Rewrite Equation [32] as

$$v' = -e^{-i\omega t} \frac{\pi A b\, e^{by}}{2M} \left[\frac{y}{i\sqrt{y^2 - \frac{x^2}{M^2}}} H_1^{(1)}\left(ib\sqrt{y^2 - \frac{x^2}{M^2}}\right) - H_o^{(1)}\left(ib\sqrt{y^2 - \frac{x^2}{M^2}}\right) \right] \qquad [33]$$

Using the following relation, (see Watson 1958)

$$K_\nu(\zeta) = \tfrac{1}{2}\pi i \; e^{\frac{1}{2}\nu\pi i} \; H_\nu^{(1)}(i\zeta)$$

where ζ is real and positive, Equation [33] becomes

$$v' = e^{-i(\omega t + \pi/2)} \frac{Ab\ e^{by}}{M} \left[\frac{y}{\sqrt{y^2 - \frac{x^2}{M^2}}} K_1\left(b\sqrt{y^2 - \frac{x^2}{M^2}}\right) + K_o\left(b\sqrt{y^2 - \frac{x^2}{M^2}}\right) \right] \qquad [34]$$

It is seen from the behaviour of K_o and K_1 that the motion in this region is a standing wave also, but, here, the amplitude of the motion is singular on the lines $y^2 = x^2/M^2$ instead of just at the origin where the source is located.

In the region

$$\frac{x^2}{M^2} - y^2 > 0$$

the arguments of the Hankel functions appearing in Equation [32] are real. From the known behaviours of the Hankel function for real arguments it is seen that internal gravity waves are present in this region.

This peculiar pattern of motion is related to the behaviour of elementary waves discussed in the previous section. The periodic source may be regarded as an integral of elementary waves of all wave numbers and of a given frequency ω. It was seen that only those waves whose number lie on a particular hyperbola are progressive. The asymptotes of the hyperbola make an angle of $\tan^{-1} M$ with the horizontal so that the direction of the wave number cannot exceed $\tan^{-1} M$. However, the group velocity or the velocity of propagation is normal to the hyperbola so that the region in which progressive waves are found are sectors bounded by the lines $y^2 = x^2/M^2$ as shown in Figure 4. Except for the small wave number range the group velocities of the elementary waves are approximately in the direction of either one of these lines. Thus, there is a large concentration of energy in them.

INTERNAL WAVES GENERATED BY AN INSTANTANEOUS SOURCE

In this section, the transient problem of the motion due to a source at the origin with a time dependency of the form $\delta(t)$ is considered. The appropriate forcing function for Equation [16] is

$$2\pi A \quad \rho_o \delta(x) \, \ddot{\delta}(t) \, [\dot{\delta}(y) - b\delta(y)]$$

Let ϕ be the solution of the equation

$$\frac{\partial^2}{\partial t^2}\left\{\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial x^2}-b^2\phi\right\}+N^2\frac{\partial^2\phi}{\partial x^2}=\delta(x)\,\delta(y)\,\dot{\delta}(t) \qquad [35]$$

therefore, the vertical velocity component v' is given by

$$v' = 2\pi A\ e^{by}\left(\frac{\partial\phi}{\partial y}-b\phi\right) \qquad [36]$$

The Fourier integral representation of ϕ is easily found to be

$$\phi(x,y,t)=\frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{e^{i(\alpha x+\beta y+\omega t)}}{\left(\frac{N^2}{\omega^2}-1\right)\alpha^2-(\beta^2+b^2)}\,d\alpha\, d\beta\, d\omega$$

[37]

Integrate first with respect to ω. The initial condition of the fluid is the state of hydrostatic equilibrium. With this consideration in mind, the path of integration is displaced slightly below the real axis. The contour is closed by a large semi-circle in the upper half plane and the integral evaluated by residues to give

$$\phi = \frac{\pi i}{(2\pi)^3} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{N\alpha\, e^{i(\alpha x+\beta y)}}{(\alpha^2+\beta^2+b^2)^{3/2}} \left(e^{-i\frac{N\alpha t}{\sqrt{\alpha^2+\beta^2+b^2}}} - e^{+i\frac{N\alpha t}{\sqrt{\alpha^2+\beta^2+b^2}}} \right) d\alpha d\beta$$

$$= \frac{\pi i}{8\pi^2} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{N\alpha\, e^{itf_1}}{(\alpha^2+\beta^2+b^2)^{3/2}}\, d\alpha d\beta \; -$$

$$\frac{\pi i}{8\pi^2} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{N\alpha\, e^{itf_2}}{(\alpha^2+\beta^2+b^2)^{3/2}}\, d\alpha d\beta = I_1 + I_2$$

where

$$f_1 = \alpha\frac{x}{t} + \beta\frac{y}{t} - \frac{N\alpha}{\sqrt{\alpha^2+\beta^2+b^2}}$$

$$f_2 = \alpha\frac{x}{t} + \beta\frac{y}{t} + \frac{N\alpha}{\sqrt{\alpha^2+\beta^2+b^2}}$$

The points of stationary phase of the integrals occur at

$$\frac{\partial f_1}{\partial \alpha} = 0, \quad \frac{\partial f_1}{\partial \beta} = 0 \qquad [38]$$

As before we consider only $x > 0$. For $x/t > 0$ only I_1 can have points of stationary phase. Given a pair $(x/t, y/t)$ the real solutions of the following simultaneous equations

$$\frac{x}{t} = \frac{N(\beta^2 + b^2)}{(\alpha^2 + B^2 + b^2)^{3/2}} \qquad [39]$$

$$\frac{y}{t} = \frac{N\alpha\beta}{(\alpha^2 + \beta^2 + b^2)^{3/2}} \qquad [40]$$

are points of stationary phase. These two equations may be combined to give

$$\left(\frac{y}{x}\right)^2 = \frac{\beta^2}{\beta^2 + b^2}\left\{\left[\frac{2gb}{\frac{x^2}{t^2}(\beta^2 + b^2)}\right]^{1/3} - 1\right\} \qquad [41]$$

For a given y/x, the above equation passes from a situation of no real solution to a situation of two real solutions as x/t tends to zero from a very large value. The transition where the equation has only one real solution, therefore, corresponds to the maximum distance from the origin at a given time where the disturbance is appreciable if we keep terms of $O(1/t)$ only. Hence, the wave front is the locus of the transition points for all y/x. The equation of the wave front is obtained by eliminating β from Equation [41] and its derivative with respect to β. The resulting equation is

$$\left(\frac{x}{t}\sqrt{\frac{2g}{b}}\right)^{2/3} = \frac{B^4}{3\left(\frac{y}{x}\right)^2 (1 + B^2)^{4/3}} \qquad [42]$$

where

$$B^2 = \frac{\sigma^2 + \sqrt{4\sigma^4 + 3\sigma^2}}{1 + \sigma^2}$$

$$\sigma = y/x$$

Since Equation [36] involves differentiation of Φ with respect to y only, the above asymptotic property of the wave front also applies to that of the internal waves generated by the instantaneous source.

Equation [42] is plotted in Figure 5. It is seen that the wave fronts for different times are geometrically similar.

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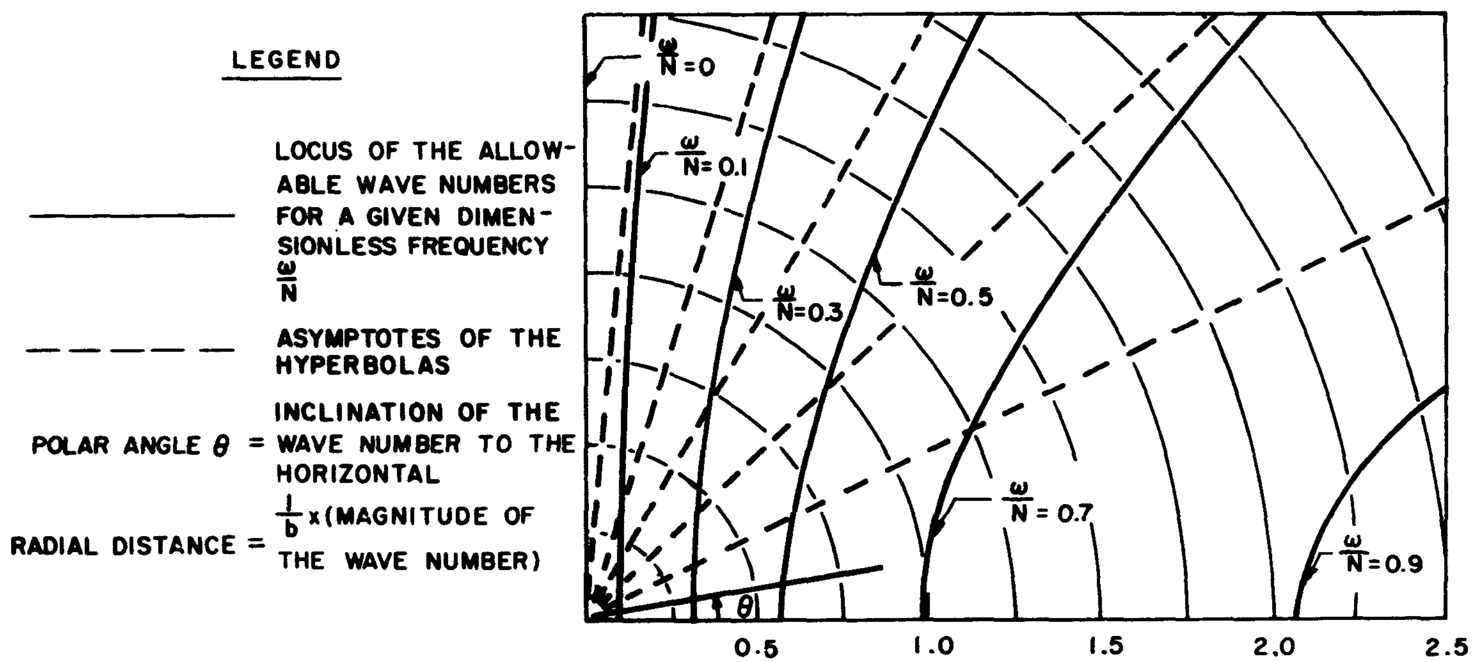
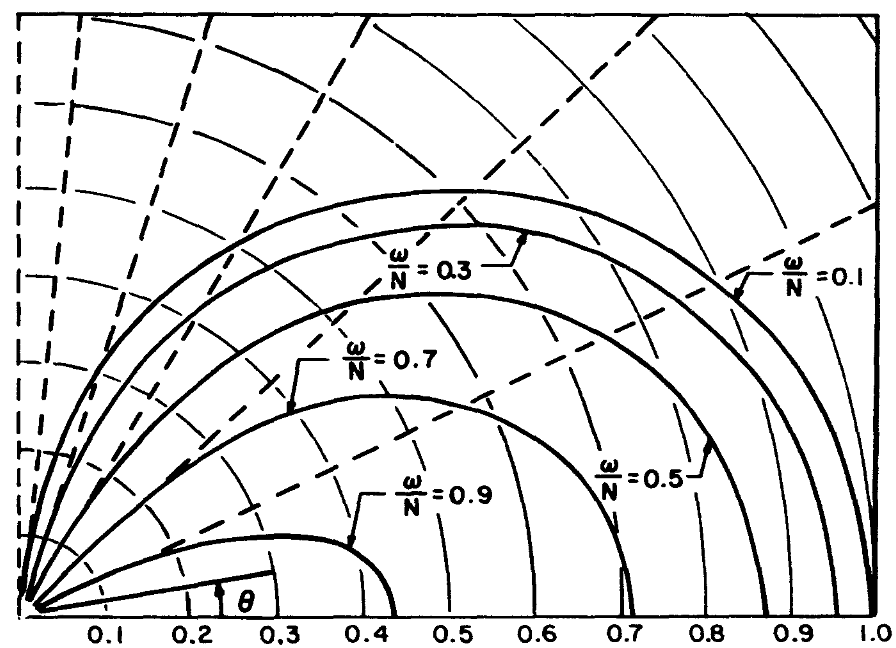


FIGURE 1 – ALLOWABLE WAVE NUMBERS FOR DIFFERENT FREQUENCIES



LEGEND

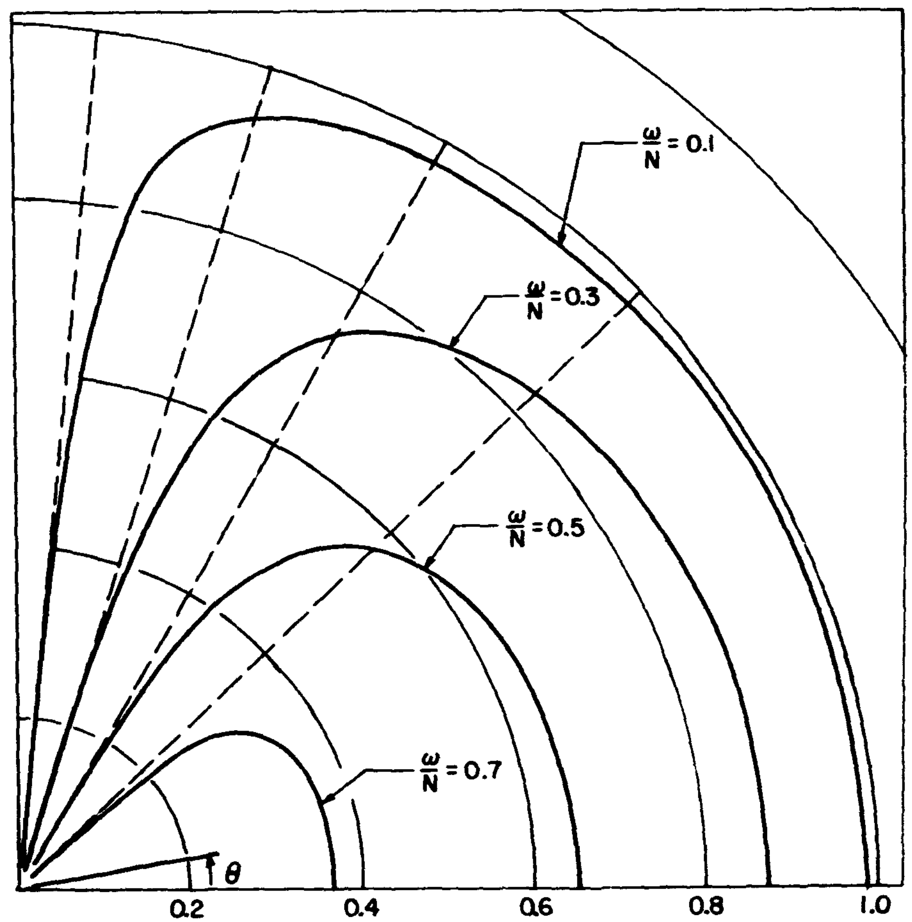
——————— THE MAGNITUDE OF THE PHASE VELOCITY FOR A GIVEN DIMENSIONLESS FREQUENCY $\frac{\omega}{N}$

– – – – – – – = MAXIMUM θ FOR A GIVEN DIMENSIONLESS FREQUENCY

POLAR ANGLE θ = INCLINATION OF THE WAVE NUMBER TO THE HORIZONTAL

RADIAL DISTANCE = $\frac{b}{N}$ x (MAGNITUDE OF THE PHASE VELOCITY)

FIGURE 2 – DEPENDENCE OF THE PHASE VELOCITY ON THE DIRECTION OF THE WAVE NUMBER



LEGEND

——— THE MAGNITUDE OF THE GROUP VELOCITY FOR A GIVEN DIMENSIONLESS FREQUENCY, $\frac{\omega}{N}$

– – – – MAXIMUM θ FOR A GIVEN DIMENSIONLESS FREQUENCY

POLAR ANGLE, θ = INCLINATION OF THE WAVE NUMBER TO THE HORIZONTAL

RADIAL DISTANCE = $\frac{b}{N}$ x (MAGNITUDE OR THE GROUP VELOCITY)

FIGURE 3 – DEPENDENCE OF THE MAGNITUDE OF THE GROUP VELOCITY ON DIRECTION OF THE WAVE NUMBER

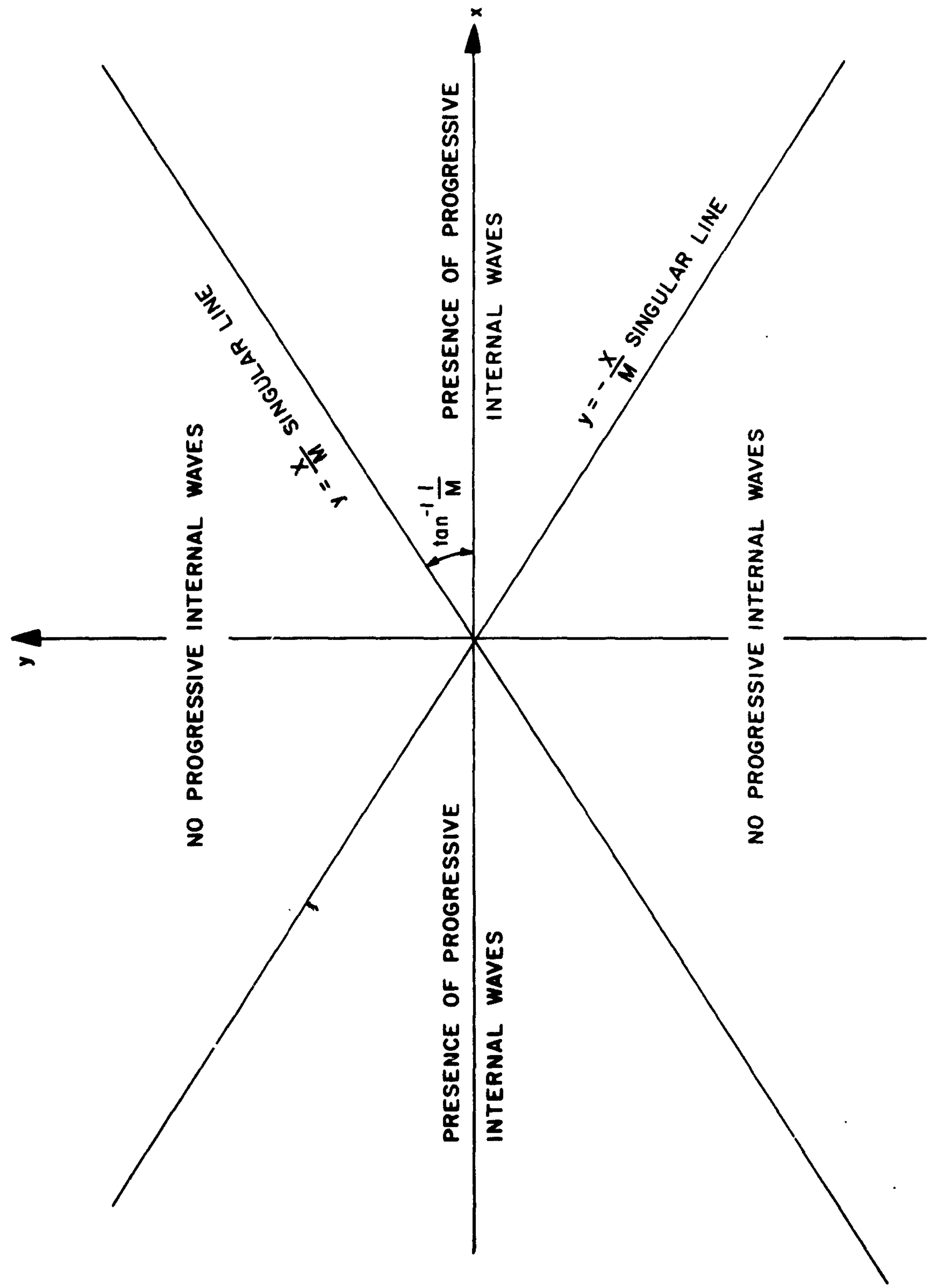


FIGURE 4 –INTERNAL MOTION DUE TO AN OSCILLATING SOURCE WHOSE FREQUENCY IS LESS THAN THE VAISALA FREQUENCY

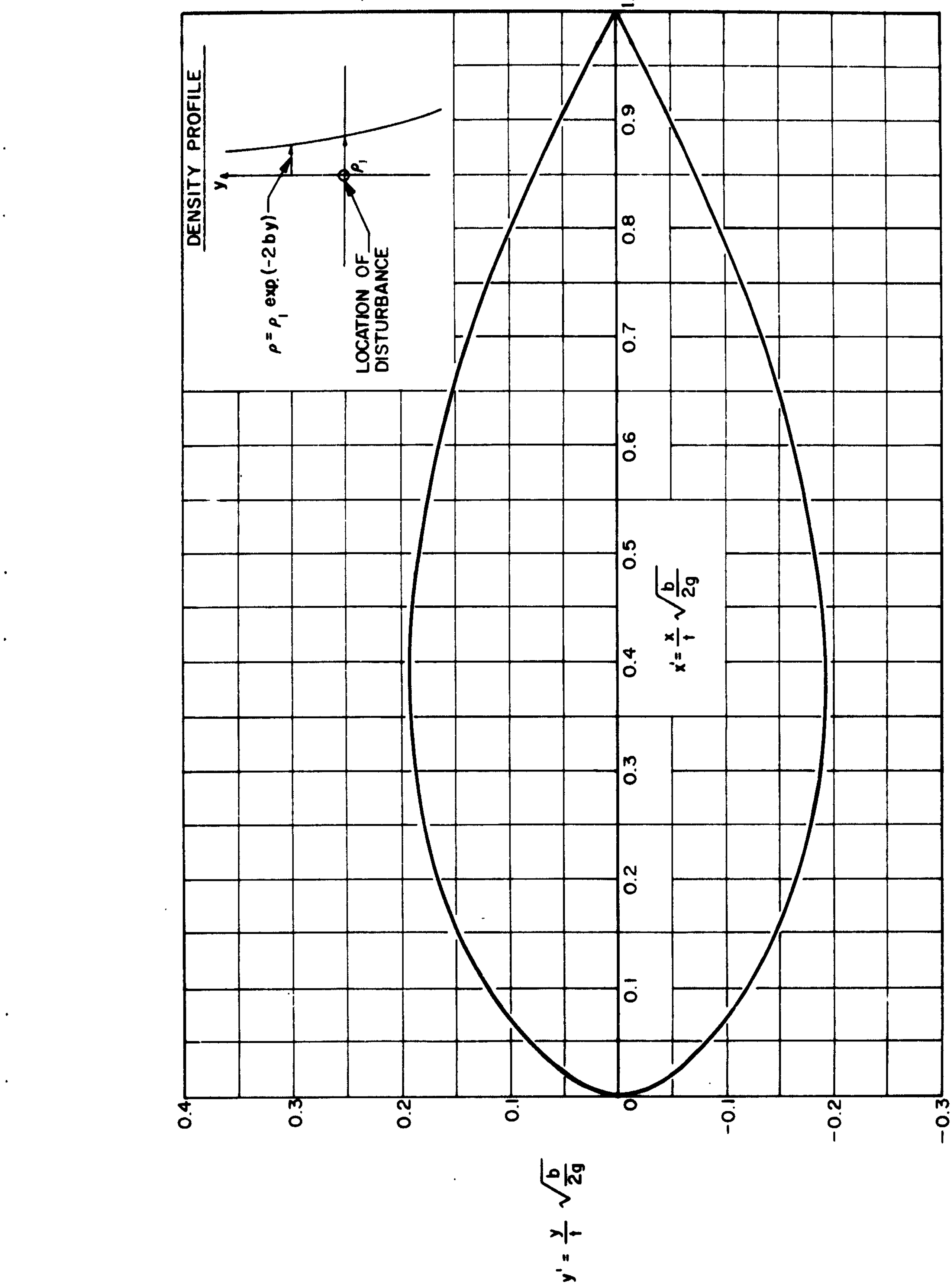


FIGURE 5–THE WAVE FRONT OF AN INSTANTANEOUS DISTURBANCE IN AN EXPONENTIALLY STRATIFIED FLUID OF INFINITE EXTENT

UNCLASSIFIED

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13. ABSTRACT

The internal waves generated by a time dependent local disturbance in an incompressible continuously stratified fluid are examined. Following the Love theory of wave motion in heterogeneous liquids, the motion is regarded as a small perturbation about the state of hydrostatic equilibrium. For simplicity, the equilibrium density profile is taken to be exponential. It was found that, for a disturbance having a simple harmonic time dependency, the ratio of the disturbance frequency to the Vaisala frequency determines whether progressive internal waves are present or not and in which regions. The propagation of the wave front of the internal waves generated by an instantaneous source was also analyzed.

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	ROLE	WT	ROLE	WT	ROLE	WT
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